# Exponential mixing for the $3D$ stochastic Navier–Stokes equations

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**Abstract**: We study the Navier-Stokes equations in dimension 3 (NS3D) driven by a noise which is white in time. We establish that if the noise is at same time sufficiently smooth and non degenerate in space, then the weak solutions converge exponentially fast to equilibrium.

We use a coupling method. The arguments used in dimension two do not apply since, as is well known, uniqueness is an open problem for NS3D. New ideas are introduced. Note however that many simplifications appears since we work with non degenerate noises.

**Key words**: Stochastic three-dimensional Navier-Stokes equations, Markov transition semi-group, invariant measure, ergodicity, coupling method, exponential mixing, galerkin approximation.

## Introduction

We are concerned with the stochastic Navier–Stokes equations on a three dimensional bounded domain (NS3D) with Dirichlet boundary conditions. These equations describe the time evolution of an incompressible fluid subjected to a determinist and a random exterior force and are given by

$$\left\{\begin{array}{l} dX + \nu(-\Delta)X\,dt + (X,\nabla)X\,dt + \nabla p\,dt = \phi(X)dW + f\,dt, \\ \begin{array}{rclll} (\mathrm{div}\ X)\,(t,\xi) & = & 0, & \text{for } \xi\in D, & t>0, \\ X(t,\xi) & = & 0, & \text{for } \xi\in \partial D, & t>0, \\ X(0,\xi) & = & x_0(\xi), & \text{for } \xi\in D. & \end{array}\end{array}\right. \tag{0.1}$$

Here $D$ is an open bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial D$ or $D=(0,1)^3$. We have denoted by $X$ the velocity, by $p$ the pressure and by $\nu$ the viscosity. The external force field acting on the fluid is the sum of a random force field of white noise type $\phi(X)dW$ and a determinist one $f\,dt$.

In the deterministic case ($\phi=0$), there exists a global weak solution (in the PDE sense) of (0.1) when $x_0$ is square integrable, but uniqueness of such solution is not known. On another hand, there exists a unique local strong solution when $x_0$ is smooth, but global existence is an open problem (see [36] for a survey on these questions).

In the stochastic case, there exists a global weak solution of the martingale problem, but pathwise uniqueness or uniqueness in law remain open problems (see [10] for a survey on the stochastic case).

The main result of the present article is to establish that, if $\phi$ is at the same time sufficiently smooth and non degenerate, then the solutions converge exponentially fast to equilibrium. More precisely, given a solution, there exists a stationary solution (which might depends on the given solution), such that the total variation distance between the laws of the given solution and of the stationary solution converges to zero exponentially fast.

Due to the lack of uniqueness, it is not straightforward to define a Markov evolution associated to (0.1). Some recent progress have been obtained in this direction. In [3], [7], under conditions on $\phi$ and $f$ very similar to ours, it is shown that every solution of (0.1) limit of Galerkin approximations verify the weak Markov property. Uniqueness in law is not known but we think that this result is a step in this direction. Our result combined with this result implies that the transition semi-group constructed in [3] is exponentially mixing.

Note also that recently, a Markov selection argument has allowed the construction of a Markov evolution in [13]. Our result does not directly apply since we only consider solutions which are limit of Galerkin approximations. However, suitable modifications of our proof might imply that under suitable assumptions on the noise, the Markov semi-group constructed in [13] is also exponentially mixing.

Our proof relies on coupling arguments. These have been introduced recently in the context of stochastic partial differential equations by several authors (see [15], [21], [24], [25], [26], [27], [30], [31] and [32]). The aim was to prove exponential mixing for degenerate noise. It was previously observed that the degeneracy of the noise on some subspace could be compensated by dissipativity arguments [1], [8], [22]. More recently, highly degenerate noise noises have been considered in [17], [28].

In all these articles, global well posedness of the stochastic equation is strongly used in many places of the proof. As already mentioned, this is not the case for the three dimensional Navier-Stokes equations considered here. Thus substantial changes in the proof have to be introduced. However, we require that the noise is sufficiently non degenerate and many difficulties of the above mentioned articles disappear.

The main idea is that coupling of solutions can be achieved for initial data which are small in a sufficiently smooth norm. A coupling satisfying good properties is constructed thanks to the Bismut-Elworthy-Li formula. Another important ingredient in our proof is that any weak solution enters a small ball in the smooth norm and that the time of entering in this ball admits an exponential moment. We overcome the lack of uniqueness of solutions by working with Galerkin approximations. We prove exponential mixing for these with constants which are controlled uniformly. Taking the limit, we obtain our result for solutions which are limit of Galerkin approximations.

## 1. Preliminaries and main result

### 1.1. **Weak solutions.**

Here $\mathcal{L}(K_1;K_2)$ (resp $\mathcal{L}_2(K_1;K_2)$) denotes the space of bounded (resp Hilbert-Schmidt) linear operators from the Hilbert space $K_1$ to $K_2$.

We denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(D;\mathbb{R}^3)$ and by $|\cdot|_p$ the norm of $L^p(D;\mathbb{R}^3)$. Recall now the definition of the Sobolev spaces $H^p(D;\mathbb{R}^3)$ for $p\in\mathbb{N}$

$$\begin{cases} H^p(D;\mathbb{R}^3)=\left\{X\in L^2(D;\mathbb{R}^3)\ \middle|\ \partial_\alpha X\in L^2(D;\mathbb{R}^3)\ \text{ for } |\alpha|\le p\right\},\\ |X|^2_{H^p}=\sum_{|\alpha|\le p}|\partial_\alpha X|^2 . \end{cases}$$

It is well known that $(H^p(D;\mathbb{R}^3),|\cdot|_{H^p})$ is a Hilbert space. The Sobolev space $H^1_0(D;\mathbb{R}^3)$ is the closure of the space of smooth functions on $D$ with compact support by $|\cdot|_{H^1}$. Setting $\|X\|=|\nabla X|$, we obtain that $\|\cdot\|$ and $|\cdot|_{H^1}$ are two equivalent norms on $H^1_0(D;\mathbb{R}^3)$ and that $(H^1_0(D;\mathbb{R}^3),\|\cdot\|)$ is a Hilbert space.

Let $H$ and $V$ be the closure of the space of smooth functions on $D$ with compact support and free divergence for the norm $|\cdot|$ and $\|\cdot\|$, respectively.
Let $\pi$ be the orthogonal projection in $L^2(D;\mathbb{R}^3)$ onto the space $H$. We set

$$A=\pi\,(-\Delta)\,,\ D(A)=V\cap H^2(D;\mathbb{R}^3),\ B(u,v)=\pi\left((u,\nabla)v\right)\ \text{ and }\ B(u)=B(u,u).$$

Let us recall the following useful identities

$$\begin{cases} (B(u,v),v) &=& 0, & u,\,v\in V,\\ (B(u,v),w) &=& -(B(u,w),v), & u,\,v,\,w\in V. \end{cases}$$

As is classical, we get rid of the pressure and rewrite problem (0.1) in the form

$$\begin{cases} dX+\nu AX\,dt+B(X)dt &=& \phi(X)dW+f\,dt,\\ X(0) &=& x_0, \end{cases} \tag{1.1}$$

where $W$ is a cylindrical Wiener process on $H$ and with a slight abuse of notations, we have denoted by the same symbols the projections of $\phi$ and $f$.

It is well-known that $(A,\mathcal{D}(A))$ is a self-adjoint operator with discrete spectrum. See [2], [34]. We consider $(e_n)_n$ an eigenbasis of $H$ associated to the increasing sequence $(\mu_n)_n$ of eigenvalues of $(A,\mathcal{D}(A))$. It will be convenient to use the fractionnal power $(A^s,\mathcal{D}(A^s))$ of the operator $(A,\mathcal{D}(A))$ for $s\in\mathbb{R}$

$$\begin{cases} \mathcal{D}(A^s) &=& \left\{X=\sum_{n=1}^\infty x_n e_n\ \middle|\ \sum_{n=1}^\infty \mu_n^{2s}|x_n|^2<\infty\right\},\\ A^sX &=& \sum_{n=1}^\infty \mu_n^s x_n e_n\ \text{ where }\ X=\sum_{n=1}^\infty x_n e_n. \end{cases}$$

We set for any $s\in\mathbb{R}$

$$\|X\|_s=\left|A^{\frac{s}{2}}X\right|,\quad \mathbb{H}_s=\mathcal{D}(A^{\frac{s}{2}}).$$

It is obvious that $(\mathbb{H}_s,\|\cdot\|_s)$ is a Hilbert space, that $(\mathbb{H}_0,\|\cdot\|_0)=(H,|\cdot|)$ and that $(\mathbb{H}_1,\|\cdot\|_1)=(V,\|\cdot\|)$. Moreover, recall that, thanks to the regularity theory of the Stokes operator, $\mathbb{H}_s$ is a closed subspace of $H^s(D,\mathbb{R}^3)$ and $\|\cdot\|_s$ is equivalent to the usual norm of $H^s(D;\mathbb{R}^3)$ when $D$ is an open bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial D$. When $D=(0,1)^3$, it remains true for $s\le 2$.

Let us define

$$\left\{\begin{array}{lcl}\mathcal{X} & = & L^\infty_{\rm loc}(\mathbb{R}^+;H)\cap L^2_{\rm loc}(\mathbb{R}^+;V)\cap C(\mathbb{R}^+;\mathbb{H}_s),\\ \mathcal{W} & = & C(\mathbb{R}^+;\mathbb{H}_{-2}),\\ \Omega_* & = & \mathcal{X}\times\mathcal{W},\end{array}\right.$$

where $s$ is any fixed negative number. Remark that the definition of $\mathcal{X}$ is not depending on $s<0$. Let $X_*$ (resp $W_*$) be the projector $\Omega_*\to\mathcal{X}$ (resp $\Omega_*\to\mathcal{W}$). The space $\Omega_*$ is endowed with its borelian $\sigma$-algebra $\mathcal{F}^*$ and with $(\mathcal{F}^*_t)_{t\geq 0}$ the filtration generated by $(X_*,W_*)$.

Recall that $W$ is said to be a $(\mathcal{F}_t)_t$–cylindrical Wiener process on $H$ if $W$ is $(\mathcal{F}_t)_t$–adapted, if $W(t+\cdot)-W(t)$ is independant of $\mathcal{F}_t$ for any $t\geq 0$ and if $W$ is a cylindrical Wiener process on $H$. Let $E$ be a Polish space. We denote by $P(E)$ the set of probability measure on $E$ endowed with the borelian $\sigma$–algebra.

**Definition 1.1** (Weak solutions)**.** *A probability measure $\mathbb{P}_\lambda$ on $(\Omega_*,\mathcal{F}^*)$ is said to be a weak solution of* (1.1) *with initial law $\lambda\in P(H)$ if the three following properties hold.*

i) *The law of $X_*(0)$ under $\mathbb{P}_\lambda$ is $\lambda$.*
ii) *The process $W_*$ is a $(\mathcal{F}^*_t)_t$–cylindrical Wiener process on $H$ under $\mathbb{P}_\lambda$.*
iii) *We have $\mathbb{P}_\lambda$-almost surely*

$$\begin{array}{l}(X_*(t),\psi)+\nu\int_0^t(X_*(s),A\psi)ds+\int_0^t(B(X_*(s)),\psi)ds\\ \qquad\qquad = (X_*(0),\psi)+t\,(f,\psi)+\int_0^t(\psi,\phi(X_*(s))dW_*(s)),\end{array}\tag{1.2}$$

*for any $t\in\mathbb{R}^+$ and any $\psi$ smooth mapping on $D$ with compact support and divergence zero.*

*When the initial value $\lambda$ is not specified, $x_0$ is the initial value of the weak solution $\mathbb{P}_{x_0}$ (i.e. $\lambda$ is equal to $\delta_{x_0}$ the Dirac mass at point $x_0$).*

These solutions are weak in both probability and PDE sense. On the one hand, these are solutions in law. Existence of solutions in law does not imply that, given a Wiener process $W$ and an initial condition $x_0$, there exist a solution $X$ associated to $W$ and $x_0$. On the other hand, these solutions live in $H$ and it is not known if they live in $\mathbb{H}_1$. This latter fact causes many problems when trying to apply Ito Formula on $F(X_*(t))$ when $F$ is a smooth mapping. Actually, we do not know if we are allowed to apply it.

That is the reason why we do not consider any weak solution but only those which are limit in distribution of solutions of Galerkin approximations of (1.1). More precisely, for any $N\in\mathbb{N}$, we denote by $P_N$ the eigenprojector of $A$ associated to the first $N$ eigenvalues. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $W$ be a cylindrical Wiener process on $H$ for $\mathbb{P}$. We consider the following approximation of (1.1)

$$\left\{\begin{array}{rcl}dX_N+\nu AX_Ndt+P_NB(X_N)dt & = & P_N\phi(X_N)dW+P_Nfdt,\\ X_N(0) & = & P_Nx_0.\end{array}\right.\tag{1.3}$$

In order to have existence of a weak solution, we use the following assumption.

**Hypothesis 1.2.** *The mapping $\phi$ is bounded Lipschitz $H\to\mathcal{L}_2(H;\mathbb{H}_1)$ and $f\in H$.*

We set

$$B_1 = \sup_{x \in H} |\phi(x)|^2_{\mathcal{L}_2(H;\mathbb{H}_1)} + \frac{|f|^2}{\nu \mu_1}.$$

It is easily shown that, given $x_0 \in H$, (1.3) has a unique solution $X_N = X_N(\cdot, x_0)$. Proceeding as in [11], we can see that the laws $(\mathbb{P}^N_{x_0})_N$ of $(X_N(\cdot, x_0), W)$ are tight in a well chosen functional space. Then, for a subsequence $(N_k)_k$, $(X_{N_k}, W)$ converges in law to $\mathbb{P}_{x_0}$ a weak solution of (1.1). Hence we have existence of the weak solutions of (1.1), but uniqueness remains an open problem.

**Remark 1.3.** *We only consider weak solutions constructed in that way. This allows to make some computations and to obtain many estimates. For instance, when trying to estimate the $L^2$-norm of $X_*(t)$ under a weak solution $\mathbb{P}_{x_0}$, we would like to apply the Ito Formula on $|X_*|^2$. This would give*

$$d |X_*|^2 + 2\nu \|X_*\|^2 dt = 2 (X_*, \phi(X_*) dW_*) + 2(f, X_*)dt + |\phi(X_*(t))|^2_{\mathcal{L}_2(H;H)} dt.$$

*Integrating and taking the expectation, we would deduce that, if $f = 0$ and $\phi$ constant,*

$$\mathbb{E}_{x_0} \left( |X_*(t)|^2 + 2\nu \int_0^t \|X_*(s)\|^2 dt \right) = |x_0|^2 + t |\phi|^2_{\mathcal{L}_2(H;H)} .$$

*Unfortunately, those computations are not allowed. However, analogous computations are valid if we replace $\mathbb{P}_{x_0}$ by $\mathbb{P}^N_{x_0}$, which yields*

$$\mathbb{E} \left( |X_N(t)|^2 + 2\nu \int_0^t \|X_N(s)\|^2 dt \right) = |P_N x_0|^2 + t |P_N \phi|^2_{\mathcal{L}_2(H;H)} .$$

*Then, we take the limit and we infer from Fatou Lemma and from the semi-continuity of $|\cdot|$, $\|\cdot\|$ in $\mathbb{H}_s$ that*

$$\mathbb{E}_{x_0} \left( |X_*(t)|^2 + 2\nu \int_0^t \|X_*(s)\|^2 dt \right) \le |x_0|^2 + t |\phi|^2_{\mathcal{L}_2(H;H)} ,$$

*provided $f = 0$ and $\phi$ constant and provided $\mathbb{P}_{x_0}$ is limit in distribution of solutions of* (1.3).

Let $\mathbb{P}'$ and $Y$ be a probability measure and a random variable on $(\Omega_*, \mathcal{F}^*)$, respectively. The distribution $\mathcal{D}_{\mathbb{P}'}(Y)$ denotes the law of $Y$ under $\mathbb{P}'$.

A weak solution $\mathbb{P}_\mu$ with initial law $\mu$ is said to be stationary if, for any $t \ge 0$, $\mu$ is equal to $\mathcal{D}_{\mathbb{P}_\mu}(X_*(t))$.

We define

$$\left(\mathcal{P}^N_t \psi\right)(x_0) = \mathbb{E}\left(\psi(X_N(t, x_0))\right) = \mathbb{E}^N_{x_0}\left(\psi(X_*(t))\right),$$

where $\mathbb{E}^N_{x_0}$ is the expectation associated to $\mathbb{P}^N_{x_0}$.

It is easily shown that $X_N(\cdot, x_0)$ verifies the strong Markov property, which obviously implies that $(\mathcal{P}^N_t)_{t \in \mathbb{R}^+}$ is a Markov transition semi-group on $P_N H$.

Ito Formula on $|X_N(\cdot, x_0)|^2$ gives

$$d |X_N|^2 + 2\nu \|X_N\|^2 dt = 2 (X_N, \phi(X_N) dW) + 2(X_N, f)dt + |P_N \phi(X_N)|^2 dt,$$

which yields, by applying arithmetico-geometric inequality and Hypothesis 1.2,

$$d |X_N|^2 + \nu \|X_N\|^2 dt \le 2 (X_N, \phi(X_N) dW) + cB_1 dt. \tag{1.4}$$

Integrating and taking the expectation, we obtain

$$\mathbb{E}\left(|X_N(t)|^2\right) \le e^{-\nu\mu_1 t}|x_0|^2 + \frac{c}{\nu\mu_1}B_1. \tag{1.5}$$

Hence, applying the Krylov-Bogoliubov Criterion (see [4]), we obtain that $(\mathcal{P}_t^N)_t$ admits an invariant measure $\mu_N$ and that every invariant measure has a moment of order two in $H$. Let $X_0^N$ be a random variable whose law is $\mu_N$ and which is independent of $W$, then $X_N = X_N(\cdot, X_0^N)$ is a stationary solution of (1.3). Integrating (1.4), we obtain

$$\mathbb{E}|X_N(t)|^2 + \nu\mathbb{E}\int_0^t \|X_N(s)\|^2 ds \le \mathbb{E}|X_N(0)|^2 + cB_1 t.$$

Since the law of $X_N(s)$ is $\mu_N$ for any $s \ge 0$ and since $\mu_N$ admits a moment of order 2, it follows

$$\int_{P_N H} \|x\|^2 \, \mu_N(dx) \le \frac{c}{\nu}B_1. \tag{1.6}$$

Moreover the laws $(\mathbb{P}_{\mu_N}^N)_N$ of $(X_N(\cdot, X_0^N), W)$ are tight in a well chosen functional space. Then, for a subsequence $(N_k')_k$, $\mathbb{P}_{\mu_{N_k}}^{N_k}$ converges in law to $\mathbb{P}_\mu$ a weak stationary solution of (1.1) with initial law $\mu$ (See [11] for details). We deduce from (1.6) that

$$\int_H \|x\|^2 \, \mu(dx) \le \frac{c}{\nu}B_1,$$

which yields (see [12])

$$\mathbb{P}_\mu\left(X_*(t) \in \mathbb{H}_1\right) = 1 \quad \text{for any} \quad t \ge 0. \tag{1.7}$$

We do not know if $X_*(t) \in \mathbb{H}_1$ for all $t$ holds $\mathbb{P}_\mu$–almost surely. This would probably imply strong uniqueness $\mu$–almost surely. Remark that it is not known in general if $\mu$ is an invariant measure because, due to the lack of uniqueness, it is not known if (1.1) defines a Markov evolution. We will see below that this is the case under suitable assumptions.

### 1.2. **Exponential convergence to equilibrium.**

In the present article, the covariance operator $\phi$ of the noise is assumed to be at the same time sufficiently smooth and non degenerate with bounded derivatives. More precisely, we use the following assumption.

**Hypothesis 1.4.** *There exist $\varepsilon > 0$ and a family $(\phi_n)_n$ of continuous mappings $H \to \mathbb{R}$ with continuous derivatives such that*

$$\begin{cases} \phi(x)dW = \sum_{n=1}^\infty \phi_n(x)e_n dW_n \quad \text{where} \quad W = \sum_{n=0}^\infty W_n e_n, \\ \kappa_0 = \sum_{n=1}^\infty \sup_{x\in H}|\phi_n(x)|^2 \mu_n^{1+\varepsilon} < \infty. \end{cases}$$

*Moreover there exists $\kappa_1$ such that for any $x, \eta \in \mathbb{H}_2$*

$$\sum_{n=1}^\infty |\phi_n'(x)\cdot\eta|^2 \mu_n^2 < \kappa_1 \|\eta\|_2^2.$$

*For any $x \in H$ and $N \in \mathbb{N}$, we have $\phi_n(x) > 0$ and*

$$\kappa_2 = \sup_{x\in H}\left|\phi^{-1}(x)\right|^2_{\mathcal{L}(\mathbb{H}_3;H)} < \infty, \tag{1.8}$$

*where*

$$\phi(x)^{-1}\cdot h=\sum_{n=1}^{\infty}\phi_n(x)^{-1}h_ne_n \quad \textit{for} \quad h=\sum_{n=0}^{\infty}h_ne_n.$$

For instance, $\phi=A^{-\frac{s}{2}}$ fulfills Hypothesis 1.4 provided $s\in\left(\frac{5}{2},3\right]$.
We set

$$B_0=\kappa_0+\kappa_1+\kappa_2+|f|^2.$$

**Remark 1.5** (Additive noise)**.** *If the noise is additive, Hypothesis 1.4 simplifies. Indeed in this case, we do not need to assume that $\phi$ and $A$ commute. This requires a different but simpler proof of Lemma 3.2 below.*

**Remark 1.6** (Large viscosity)**.** *Another situation where we can get rid of the assumption that the noise is diagonal is when the viscosity $\nu$ is sufficiently large. The proof is simpler in that case.*

**Remark 1.7.** *It is easily shown that Hypothesis 1.4 and $f\in H$ imply Hypothesis 1.2. Therefore, solutions of* (1.3) *are well-defined and, for a subsequence, they converge to weak solution of* (1.1)*.*

The aim of the present article is to establish that, under Hypothesis 1.4 and under a condition of smallness of $\|f\|_\varepsilon$, the law of $X_*(t)$ under a weak solution $\mathbb{P}_{x_0}$ converges exponentially fast to equilibrium provided $\mathbb{P}_{x_0}$ is limit in distribution of solutions of (1.3).

Before stating our main result, let us recall some definitions. Let $E$ be Polish space. The set of all probability measures on $E$ is denoted by $\mathcal{P}(E)$. The set of all bounded measurable (resp uniformly continuous) maps from $E$ to $\mathbb{R}$ is denoted by $B_b(E;\mathbb{R})$ (resp $UC_b(E;\mathbb{R})$). The total variation $\|\mu\|_{var}$ of a finite real measure $\lambda$ on $E$ is given by

$$\|\lambda\|_{var}=\sup\{|\lambda(\Gamma)| \mid \Gamma\in\mathcal{B}(E)\},$$

where we denote by $\mathcal{B}(E)$ the set of the Borelian subsets of $E$.

The main result of the present article is the following. Its proof is given in section 4 after several preliminary results.

**Theorem 1.8.** *Assume that Hypothesis 1.4 holds. There exists $\delta^0$, $C$ and $\gamma>0$ only depending on $\phi$, $D$, $\varepsilon$ and $\nu$ such that, for any weak solution $\mathbb{P}_\lambda$ with initial law $\lambda\in\mathcal{P}(H)$ which is limit of solutions of* (1.3)*, there exists a weak stationary solution $\mathbb{P}_\mu$ with initial law $\mu$ such that*

$$\|\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t))-\mu\|_{var}\le Ce^{-\gamma t}\left(1+\int_H|x|^2\,\lambda(dx)\right), \tag{1.9}$$

*provided $\|f\|_\varepsilon^2\le\delta^0$ and where $\|\cdot\|_{var}$ is the total variation norm associated to the space $\mathbb{H}_s$ for $s<0$.*

*Moreover, for a given $\mathbb{P}_\lambda$, $\mu$ is unique and $\mathbb{P}_\mu$ is limit of solutions of* (1.3)*.*

It is well known that $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$ which means that for any finite measure $\lambda'$ on $\mathbb{H}_s$ for $s<0$

$$\|\lambda'\|_{var}=\sup_{|g|_\infty\le 1}\left|\int_{\mathbb{H}_s}g(x)\,\lambda'(dx)\right|,$$

where the supremum is taken over $g \in UC_b(\mathbb{H}_s)$ which verifies $|g|_\infty \le 1$. Hence (1.9) is equivalent to

$$\left|\mathbb{E}_\lambda \left(g(X_*(t))\right) - \int_H g(x)\,\mu(dx)\right| \le C\,|g|_\infty \left(1 + \int_H |x|^2\,\lambda(dx)\right), \tag{1.10}$$

for any $g \in UC_b(\mathbb{H}_s)$.

**Remark 1.9** (Topology associated to the total variation norm)**.** *Remark that if $\lambda'$ is a finite measure of $\mathbb{H}_{s_0}$, then the value of the total variation norm of $\lambda'$ associated to the space $\mathbb{H}_s$ is not depending of the value of $s \le s_0$.*

*Hence, since $\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t))$ is a probability measure on $H$ then* (1.9) *(resp* (1.10)*) remains true when $\|\cdot\|_{var}$ is the total variation norm associated to the space $H$ (resp for any $g \in B_b(H;\mathbb{R})$).*

*Moreover, we see below that, under suitable assumptions, if $\lambda$ is a probability measure on $\mathbb{H}_2$, then $\mathcal{D}_{\mathbb{P}_\lambda}(X_*(t))$ is still a probability measure on $\mathbb{H}_2$. It follows that* (1.9) *(resp* (1.10)*) remains true when $\|\cdot\|_{var}$ is associated to $\mathbb{H}_2$ (resp for any $g \in B_b(\mathbb{H}_2;\mathbb{R})$).*

Our method is not influenced by the size of the viscosity $\nu$. Then, for simplicity in the redaction, we now assume that $\nu = 1$.

### 1.3. **Markov evolution.**

Here, we take into account the results of [3], [7] and we rewrite Theorem 1.8. This section is not necessary in the understanding of the proof of Theorem 1.8.

Let $(N'_k)_k$ be an increasing sequence of integer. In [3], [7], it is established that it is possible to extract a subsequence $(N_k)_k$ of $(N'_k)_k$ such that, for any $x_0 \in \mathbb{H}_2$, $\mathbb{P}_{x_0}^{N_k}$ converges in distribution to a weak solution $\mathbb{P}_{x_0}$ of (1.1) provided the following assumption holds.

**Hypothesis 1.10.** *There exist $\varepsilon, \delta > 0$ such that the mapping $\phi$ is bounded in $\mathcal{L}_2(H;\mathbb{H}_{1+\varepsilon})$. Moreover, for any $x$, ker $\phi(x) = \{0\}$ and there exits a bounded map $\phi^{-1} : H \to \mathcal{L}(\mathbb{H}_{3-\delta};H)$ such that for any $x \in H$,*

$$\phi(x)\cdot\phi^{-1}(x)\cdot h = h \quad \textit{for any } h \in \mathbb{H}_{3-\delta}.$$

*Moreover $f \in V$.*

The method to extract $(N_k)_k$ is based on the investigation of the properties of the Kolmogorov equation associated to (1.1) perturbed by a very irregular potential.

It follows that $(\mathbb{P}_{x_0})_{x_0\in\mathbb{H}_2}$ is a weak Markov family, which means that for any $x_0 \in \mathbb{H}_2$

$$\mathbb{P}_{x_0}\left(X_*(t) \in \mathbb{H}_2\right) = 1 \quad \text{for any } t \ge 0. \tag{1.11}$$

and that, for any $t_1 < \cdots < t_n$, $t > 0$ and any $\psi \in B_b(\mathbb{H}_2;\mathbb{R})$

$$\mathbb{E}_{x_0}\left(\psi(X_*(t+t_n))\,|\,X_*(t_1),\dots,X_*(t_n)\right) = \mathcal{P}_t\psi(X_*(t_n)), \tag{1.12}$$

where

$$\left(\mathcal{P}_t\psi\right)(x_0) = \mathbb{E}_{x_0}\left(\psi(X_*(t))\right).$$

Note that (1.11) was known only for a stationary solution (see [12]).

**Remark 1.11.** *Assume that Hypothesis 1.4 holds. If we strengthen* (1.8) *into*

$$\kappa_2 = \sup_{x\in H}\left|\phi^{-1}(x)\right|^2_{\mathcal{L}(\mathbb{H}_{3-\delta};H)} < \infty,$$

*for some $\delta > 0$, then Hypothesis* (1.10) *holds.*

Hence, we immediately deduce the following corollary from Theorem 1.8.

**Corollary 1.12.** *Assume that Hypothesis 1.4 and 1.10 hold. Then there exits a unique invariant measure $\mu$ for $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ and $C,\gamma>0$ such that for any $\lambda\in\mathcal{P}(\mathbb{H}_2)$*

$$\|\mathcal{P}_t^*\lambda-\mu\|_{var}\le Ce^{-\gamma t}\left(1+\int_{\mathbb{H}_2}|x|^2\ \lambda(dx)\right), \tag{1.13}$$

*provided $\|f\|_\varepsilon^2\le\delta^0$ and where $\|\cdot\|_{var}$ is the total variation norm associated to the space $\mathbb{H}_2$.*

**Remark 1.13** (Uniqueness of the invariant measure $\mu$)**.** *Assume that Hypothesis 1.10 holds. Let $\mathbb{P}_{x_0}$ and $\mathbb{P}'_{x_0}$ be two weak solutions of* (1.1) *which are limit in distribution of solutions of* (1.3)*. Then we build $(\mathcal{P}_t)_t$ and $(\mathcal{P}'_t)_t$ as above associated to $\mathbb{P}_{x_0}$ and $\mathbb{P}'_{x_0}$, respectively. It follows that there exists $\mu$ and $\mu'$ such that* (1.13) *and* (1.10) *hold for $((\mathcal{P}_t)_t,\mathbb{P}_{x_0},\mu)$ and $((\mathcal{P}'_t)_t,\mathbb{P}'_{x_0},\mu')$. Although we have uniqueness of the invariant measures $\mu$ and $\mu'$ associated to $(\mathcal{P}_t)_t$ and $(\mathcal{P}'_t)_t$, we do not know if $\mu$ and $\mu'$ are equal.*

1.4. **Coupling methods.**

The proof of Theorem 1.8 is based on coupling arguments. We now recall some basic results about coupling. Moreover, in order to explain the coupling method in the case of non degenerate noise, we briefly give the proof of exponential mixing for equation (1.3).

Let $(\lambda_1,\lambda_2)$ be two distributions on a polish space $(E,\mathcal{E})$ and let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $(Z_1,Z_2)$ be two random variables $(\Omega,\mathcal{F})\to(E,\mathcal{E})$. We say that $(Z_1,Z_2)$ is a coupling of $(\lambda_1,\lambda_2)$ if $\lambda_i=\mathcal{D}(Z_i)$ for $i=1,2$. We have denoted by $\mathcal{D}(Z_i)$ the law of the random variable $Z_i$.

Next result is fundamental in the coupling methods, the proof is given for instance in the Appendix of [30].

**Lemma 1.14.** *Let $(\lambda_1,\lambda_2)$ be two probability measures on $(E,\mathcal{E})$. Then*

$$\|\lambda_1-\lambda_2\|_{var}=\min\mathbb{P}(Z_1\neq Z_2).$$

*The minimum is taken over all couplings $(Z_1,Z_2)$ of $(\lambda_1,\lambda_2)$. There exists a coupling which reaches the minimum value. It is called a maximal coupling.*

Let us first consider the case of the solutions of (1.3). Assume that Hypothesis 1.4 holds. Let $N\in\mathbb{N}$ and $(x_0^1,x_0^2)\in\mathbb{R}^2$. Combining arguments from [23], [27], it can be shown that there exists a decreasing function $p_N(\cdot)>0$ such that

$$\left\|\left(\mathcal{P}_1^N\right)^*\delta_{x_0^2}-\left(\mathcal{P}_1^N\right)^*\delta_{x_0^1}\right\|_{var}\le 1-p_N\left(\left|x_0^1\right|+\left|x_0^2\right|\right). \tag{1.14}$$

Applying Lemma 1.14, we build a maximal coupling $(Z_1,Z_2)=(Z_1(x_0^1,x_0^2),Z_2(x_0^1,x_0^2))$ of $(\left(\mathcal{P}_1^N\right)^*\delta_{x_0^1},\left(\mathcal{P}_1^N\right)^*\delta_{x_0^2})$. It follows

$$\mathbb{P}\left(Z_1=Z_2\right)\ge p_N\left(\left|x_0^1\right|+\left|x_0^2\right|\right)>0. \tag{1.15}$$

Let $(W,\widetilde{W})$ be a a couple of independent cylindrical Wiener processes and $\delta>0$. We denote by $X_N(\cdot,x_0)$ and $\widetilde{X}_N(\cdot,x_0)$ the solutions of (1.3) associated to $W$ and $\widetilde{W}$, respectively. Now we build a couple of random variables $(V_1,V_2)=$

$(V_1(x_0^1,x_0^2),V_2(x_0^1,x_0^2))$ on $P_NH$ as follows

$$(1.16)\quad (V_1,V_2)=\begin{cases}(X_N(\cdot,x_0),X_N(\cdot,x_0)) & \text{if } x_0^1=x_0^2=x_0,\\ (Z_1(x_0^1,x_0^2),Z_2(x_0^1,x_0^2)) & \text{if } (x_0^1,x_0^2)\in B_H(0,\delta)\backslash\{x_0^1=x_0^2\},\\ (X_N(\cdot,x_0^1),\widetilde{X}_N(\cdot,x_0^2)) & \text{else,}\end{cases}$$

where $B_H(0,\delta)$ is the ball of $H\times H$ with radius $\delta$.

Then $(V_1(x_0^1,x_0^2),V_2(x_0^1,x_0^2))$ is a coupling of $((\mathcal{P}_1^N)^*\,\delta_{x_0^1},(\mathcal{P}_1^N)^*\,\delta_{x_0^2})$. It can be shown that it depends measurably on $(x_0^1,x_0^2)$. We then build a coupling $(X^1,X^2)$ of $(\mathcal{D}(X_N(\cdot,x_0^1)),\mathcal{D}(X_N(\cdot,x_0^2)))$ by induction on $\mathbb{N}$. We first set $X^i(0)=x_0^i$ for $i=1,2$. Then, assuming that we have built $(X^1,X^2)$ on $\{0,1,\dots,k\}$, we take $(V_1,V_2)$ as above independent of $(X^1,X^2)$ and set

$$X^i(k+1)=V_i(X^1(k),X^2(k))\qquad\text{for } i=1,2.$$

Taking into account (1.5), it is easily shown that the time of return of $(X^1,X^2)$ in $B(0,4(c/\mu_1)B_1)$ admits an exponential moment. We choose $\delta=4(c/\mu_1)B_1$. It follows from (1.15), (1.16) that, $(X^1(n),X^2(n))\in B(0,\delta)$ implies that the probability of having $(X^1,X^2)$ coupled (i.e. equal) at time $n+1$ is bounded below by $p_N(2\delta)>0$. Finally, remark that if $(X^1,X^2)$ are coupled at time $n+1$, then they remain coupled for any time after. Combining these three properties and using the fact that $(X^1(n),X^2(n))_{n\in\mathbb{N}}$ is a discrete strong Markov process, it is easily shown that

$$(1.17)\qquad \mathbb{P}\left(X^1(n)\neq X^2(n)\right)\le C_Ne^{-\gamma_N n}\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right),$$

with $\gamma_N>0$.

Recall that $(X^1,X^2)$ is a coupling of $(\mathcal{D}(X_N(\cdot,x_0^1)),\mathcal{D}(X_N(\cdot,x_0^2)))$ on $\mathbb{N}$. It follows that $(X^1(n),X^2(n))$ is a coupling of $((\mathcal{P}_n^N)^*\delta_{x_0^1},(\mathcal{P}_n^N)^*\delta_{x_0^2})$. Combining Lemma 1.14 and (1.17), we obtain, for $n\in\mathbb{N}$,

$$\left\|\left(\mathcal{P}_n^N\right)^*\delta_{x_0^2}-\left(\mathcal{P}_n^N\right)^*\delta_{x_0^1}\right\|_{var}\le C_Ne^{-\gamma_N n}\left(1+\left|x_0^1\right|^2+\left|x_0^2\right|^2\right).$$

Setting $n=\lfloor t\rfloor$ and integrating $(x_0^2,x_0^1)$ over $((\mathcal{P}_{t-n}^N)^*\lambda)\otimes\mu_N$ where $\mu_N$ is an invariant measure, it follows that, for any $\lambda\in P(P_NH)$,

$$(1.18)\qquad \left\|\left(\mathcal{P}_t^N\right)^*\lambda-\mu_N\right\|_{var}\le C_Ne^{-\gamma_N t}\left(1+\int_{P_NH}|x|^2\,\lambda(dx)\right).$$

This result is useless when considering equation (1.1) since the constants $C_N$, $\gamma_N$ strongly depend on $N$. If one tries to apply directly the above arguments to the infinite dimensional equation (1.1), one faces several difficulties. First it is not known whether $\mathbb{P}_{x_0}$ is Markov. We only know that, as explained in section 1.3, a Markov transition semi-group can be constructed. This is a major difficulty since this property is implicitely used in numerous places above. Another strong problem is that Girsanov transform is used in order to obtain (1.14). Contrary to the two dimensional case, no Foias-Prodi estimate is available for the three dimensional Navier-Stokes equations and the Girsanov transform should be done in the infinite dimensional equation. This seems impossible. We will show that we are able to prove an analogous result to (1.14) by a completely different argument. However, this will hold only for small initial data in $\mathbb{H}_2$. Another problem will occur since it is not known whether solutions starting in $\mathbb{H}_2$ remain in $\mathbb{H}_2$.

We remedy the lack of Markov property by working only on Galerkin approximations and prove that (1.18) holds with constants uniform in $N$. As already mentioned, we prove that (1.14) is true for $x_0^1$, $x_0^2$ in a small ball of $\mathbb{H}_2$ and uniformly in $N$. Then, following the above argument, it remains to prove that the time of return in this small ball admits an exponential moment. Note that the smallness assumption on $f$ is used at this step. In the following sections, we prove

**Proposition 1.15.** *Assume that Hypothesis 1.4 holds. Then there exist $\delta^0 = \delta^0(B_0, D, \varepsilon, \nu)$, $C = C(\phi, D, \varepsilon, \nu) > 0$ and $\gamma = \gamma(\phi, D, \varepsilon, \nu) > 0$ such that if $\|f\|_\varepsilon^2 \le \delta^0$ holds, then, for any $N \in \mathbb{N}$, there exists a unique invariant measure $\mu_N$ for $\left(\mathcal{P}_t^N\right)_{t\in\mathbb{R}^+}$. Moreover, for any $\lambda \in \mathcal{P}(P_N H)$*

$$\left\|\left(\mathcal{P}_t^N\right)^* \lambda - \mu_N\right\|_{var} \le Ce^{-\gamma t}\left(1 + \int_{P_N H} |x|^2\, \lambda(dx)\right). \tag{1.19}$$

We now explain why this result implies Theorem 1.8.

Let $\lambda \in P(H)$ and $X_\lambda$ be a random variable on $H$ whose law is $\lambda$ and which is independant of $W$. Since $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$, then (1.19) implies that

$$\left|\mathbb{E}\left(g(X_N(t, X_\lambda))\right) - \int_{P_N H} g(x)\, \mu_N(dx)\right| \le C\,|g|_\infty \left(1 + \int_H |x|^2\, \lambda(dx)\right), \tag{1.20}$$

for any $g \in UC_b(\mathbb{H}_s)$ for $s < 0$.

Assume that, for a subsequence $(N_k')_k$, $X_N(t, X_\lambda)$ converges in distribution in $\mathbb{H}_s$ to the law $X_*(t)$ under the weak solution $\mathbb{P}_\lambda$ of (1.1). Recall that the family $(\mathbb{P}_{\mu_N}^N)_N$ is tight. Hence, for a subsequence $(N_k)_k$ of $(N_k')_k$, $\mathbb{P}_{\mu_{N_k}}$ converges to $\mathbb{P}_\mu$ a weak stationary solution of (1.1) with initial law $\mu$. Taking the limit, (1.10) follows from (1.20), which yields Theorem 1.8.

## 2. Coupling of solutions starting from small initial data

The aim of this section is to establish the following result. A result analogous to (1.15) but uniform in $N$.

**Proposition 2.1.** *Assume that Hypothesis 1.4 holds and that $f \in H$. Then there exist $(T, \delta) \in (0, 1)^2$ such that, for any $N \in \mathbb{N}$, there exists a coupling $(Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2))$ of $((\mathcal{P}_T^N)^*\delta_{x_0^1}, (\mathcal{P}_T^N)^*\delta_{x_0^2})$ which measurably depends on $(x_0^1, x_0^2) \in \mathbb{H}_2$ and which verifies*

$$\mathbb{P}\left(Z_1(x_0^1, x_0^2) = Z_2(x_0^1, x_0^2)\right) \ge \frac{3}{4} \tag{2.1}$$

*provided*

$$\left\|x_0^1\right\|_2^2 \vee \left\|x_0^2\right\|_2^2 \le \delta. \tag{2.2}$$

Assume that Hypothesis 1.4 holds and that $f \in H$. Let $T \in (0, 1)$. Applying Lemma 1.14, we build $(Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2))$ as the maximal coupling of $(\mathcal{P}_T^*\delta_{x_0^1}, \mathcal{P}_T^*\delta_{x_0^2})$. Measurable dependance follows from a slight extension of Lemma 1.17 (see [30], remark $A.1$).

In order to establish Proposition 2.1, it is sufficient to prove that there exists $c(B_0, D)$ not depending on $T \in (0, 1)$ and on $N \in \mathbb{N}$ such that

$$\left\|\left(\mathcal{P}_T^N\right)^* \delta_{x_0^2} - \left(\mathcal{P}_T^N\right)^* \delta_{x_0^1}\right\|_{var} \le c(B_0, D)\sqrt{T}, \tag{2.3}$$

provided

$$\left\|x_0^1\right\|_2^2 \vee \left\|x_0^2\right\|_2^2 \leq B_0 T^3. \tag{2.4}$$

Then it suffices to choose $T \leq 1/(4c(B_0, D))^2$ and $\delta = B_0 T^3$.

Since $\|\cdot\|_{var}$ is the dual norm of $|\cdot|_\infty$, (2.3) is equivalent to

$$\left|\mathbb{E}\left(g(X_N(T, x_0^2)) - g(X_N(T, x_0^1))\right)\right| \leq 8\, |g|_\infty\, c(B_0, D)\sqrt{T}. \tag{2.5}$$

for any $g \in UC_b(P_N H)$.

It follows from the density of $C_b^1(P_N H) \subset UC_b(P_N H)$ that, in order to establish Proposition 2.1, it is sufficient to prove that (2.5) holds for any $N \in \mathbb{N}$, $T \in (0, 1)$ and $g \in C_b^1(P_N H)$ provided (2.4) holds.

The proof of (2.5) under this condition is splitted into the next three subsections.

### 2.1. **A priori estimate.**

For any process $X$, we define the $\mathbb{H}_1$–energy of $X$ at time $t$ by

$$E_X^{\mathbb{H}_1}(t) = \|X(t)\|^2 + \int_0^t \|X(s)\|_2^2\, ds.$$

Now we establish the following result which will be useful in the proof of 2.5.

**Lemma 2.2.** *Assume that Hypothesis 1.4 holds and that $f \in H$. There exist $K_0 = K_0(D)$ and $c = c(D)$ such that for any $T \leq 1$ and any $N \in \mathbb{N}$, we have*

$$\mathbb{P}\left(\sup_{(0,T)} E_{X_N(\cdot, x_0)}^{\mathbb{H}_1} > K_0\right) \leq c\left(1 + \frac{B_0}{K_0}\right)\sqrt{T},$$

*provided $\|x_0\|^2 \leq B_0 T$.*

Let $X_N = X_N(\cdot, x_0)$. Ito Formula on $\|X_N\|^2$ gives

$$d\|X_N\|^2 + 2\|X_N\|_2^2\, dt = dM_{\mathbb{H}_1} + I_{\mathbb{H}_1} dt + \|P_N \phi(X_N)\|_{\mathcal{L}_2(H;\mathbb{H}_1)}^2\, dt + I_f dt, \tag{2.6}$$

where

$$\begin{cases} I_{\mathbb{H}_1} = -2\left(AX_N, B(X_N)\right),\ I_f = 2\left(AX_N, f\right), \\ M_{\mathbb{H}_1}(t) = 2\int_0^t \left(AX_N(s), \phi(X_N(s)) dW(s)\right). \end{cases}$$

Combining a Hölder inequality, a Agmon inequality and a arithmetico-geometric inequality gives

$$I_{\mathbb{H}_1} \leq 2\|X_N\|_2 |X_N|_\infty \|X_N\| \leq c\|X_N\|_2^{\frac{3}{2}} \|X_N\|^{\frac{3}{2}} \leq \frac{1}{4}\|X_N\|_2^2 + c\|X_N\|^6. \tag{2.7}$$

Similarly, using Poincaré inequality and Hypothesis 1.4,

$$I_f \leq \frac{1}{4}\|X_N\|_2^2 + c|f|^2 \leq \frac{1}{4}\|X_N\|_2^2 + cB_0. \tag{2.8}$$

We deduce from (2.6), (2.7), (2.8), Hypothesis 1.4 and Poincaré inequality that

$$d\|X_N\|^2 + \|X_N\|_2^2\, dt \leq dM_{\mathbb{H}_1} + cB_0 dt + c\|X_N\|^2\left(\|X_N\|^4 - 4K_0^2\right) dt, \tag{2.9}$$

where

$$K_0 = \sqrt{\frac{\mu_1}{8c}}. \tag{2.10}$$

Setting

$$\sigma_{\mathbb{H}_1} = \inf\left\{ t \in (0,T) \;\middle|\; \|X_N(t)\|^2 > 2K_0 \right\},$$

we infer from $\|x_0\|^2 \le B_0 T$ that for any $t \in (0, \sigma_{\mathbb{H}_1})$

$$E^{\mathbb{H}_1}_{X_N}(t) \le cB_0T + M_{\mathbb{H}_1}(t). \tag{2.11}$$

We deduce from Hypothesis 1.4 and from Poincaré inequality that $\phi(x)^*A$ is bounded in $\mathcal{L}(\mathbb{H}_1;\mathbb{H}_1)$ by $cB_0$. It follows that for any $t \in (0, \sigma_{\mathbb{H}_1})$

$$\langle M_{\mathbb{H}_1}\rangle(t) = 4\int_0^t \|P_N\phi(X_N(s))^*AX_N(s)\|^2\,dt \le cB_0\int_0^t \|X_N(s)\|^2\,ds \le 2cK_0B_0T.$$

Hence a Burkholder-Davis-Gundy inequality gives

$$\mathbb{E}\left(\sup_{(0,\sigma_{\mathbb{H}_1})} M_{\mathbb{H}_1}\right) \le c\mathbb{E}\sqrt{\langle M_{\mathbb{H}_1}\rangle(\sigma_{\mathbb{H}_1})} \le c\sqrt{K_0B_0T} \le c(K_0+B_0)\sqrt{T}.$$

It follows from (2.11) and $T \le 1$ that

$$\mathbb{E}\left(\sup_{(0,\sigma_{\mathbb{H}_1})} E^{\mathbb{H}_1}_{X_N}\right) \le c(B_0+K_0)\sqrt{T},$$

which yields, by a Chebyshev inequality,

$$\mathbb{P}\left(\sup_{(0,\sigma_{\mathbb{H}_1})} E^{\mathbb{H}_1}_{X_N} > K_0\right) \le c\left(1+\frac{B_0}{K_0}\right)\sqrt{T}.$$

Now, since $\sup_{(0,\sigma_{\mathbb{H}_1})} E^{\mathbb{H}_1}_{X_N} \le K_0$ implies $\sigma_{\mathbb{H}_1} = T$, we deduce Lemma 2.2.

2.2. **Estimate of the derivative of $X_N$.**

Let $N \in \mathbb{N}$ and $(x_0,h) \in (\mathbb{H}_2)^2$. We are concerned with the following equation

$$\left\{\begin{array}{rcl} d\eta_N + A\eta_N\,dt + P_N\widetilde{B}(X_N,\eta_N)\,dt &=& P_N(\phi'(X_N)\cdot\eta_N)\,dW, \\ \eta_N(s,s,x_0)\cdot h &=& P_Nh, \end{array}\right. \tag{2.12}$$

where $\widetilde{B}(X_N,\eta_N) = B(X_N,\eta_N) + B(\eta_N,X_N)$, $X_N = X_N(\cdot,x_0)$ and $\eta_N(t) = \eta_N(t,s,x_0)\cdot h$ for $t \ge s$.

Existence and uniqueness of the solutions of (2.12) are easily shown. Moreover if $g \in C^1_b(P_NH)$, then, for any $t \ge 0$, we have

$$\left(\nabla\left(\mathcal{P}^N_t g\right)(x_0), h\right) = \mathbb{E}\left(\nabla g(X_N(t,x_0)), \eta_N(t,0,x_0)\cdot h\right). \tag{2.13}$$

For any process $X$, we set

$$\sigma(X) = \inf\left\{ t \in (0,T) \;\middle|\; \int_0^t \|X(s)\|_2^2\,ds \ge K_0+1 \right\}, \tag{2.14}$$

where $K_0$ is defined in Lemma 2.2. We establish the following result.

**Lemma 2.3.** *Assume that Hypothesis 1.4 holds and that $f \in H$. Then there exists $c = c(B_0,D)$ such that for any $N \in \mathbb{N}$, $T \le 1$ and $(x_0,h) \in (\mathbb{H}_2)^2$*

$$\mathbb{E}\int_0^{\sigma(X_N(\cdot,x_0))} \|\eta_N(t,0,x_0)\cdot h\|_3^2\,dt \le c\,\|h\|_2^2.$$

For a better readability, we set $\eta_N(t) = \eta_N(t, 0, x_0)\cdot h$ and $\sigma = \sigma(X_N(\cdot, x_0))$. Ito Formula on $\|\eta_N(t)\|_2^2$ gives

$$(2.15)\quad d\|\eta_N\|_2^2 + 2\|\eta_N\|_3^2\,dt = dM_{\eta_N} + I_{\eta_N}\,dt + \|P_N\left(\phi'(X_N)\cdot\eta_N\right)\|^2_{\mathcal{L}_2(U;\mathbb{H}_2)}\,dt,$$

where

$$\begin{cases} M_{\eta_N}(t) &=& 2\int_0^t \left(A^2\eta_N, \left(P_N\phi'(X_N)\cdot\eta_N\right)dW\right)\,ds,\\ I_{\eta_N} &=& -2\left(A^{\frac{3}{2}}\eta_N, A^{\frac{1}{2}}\widetilde{B}(X_N,\eta_N)\right).\end{cases}$$

It follows from Hölder inequalities, Sobolev Embedding and a arithmetico-geometric inequality

$$I_{\eta_N} \le c\|\eta_N\|_3\|\eta_N\|_2\|X_N\|_2 \le \|\eta_N\|_3^2 + c\|\eta_N\|_2^2\|X_N\|_2^2.$$

Hence, we deduce from (2.15) and Hypothesis 1.4

$$d\|\eta_N\|_2^2 + \|\eta_N\|_3^2\,dt \le dM_{\eta_N} + c\|\eta_N\|_2^2\|X_N\|_2^2 + B_0\|\eta_N\|_2^2\,dt.$$

Integrating and taking the expectation, we obtain

$$(2.16)\qquad \mathbb{E}\left(\mathcal{E}(\sigma,0)\|\eta_N(\sigma)\|_2^2 + \int_0^\sigma \mathcal{E}(\sigma,t)\|\eta_N(t)\|_3^2\,dt\right) \le \|h\|_2^2,$$

where

$$\mathcal{E}(t,s) = e^{-B_0t - c\int_s^t\|X_N(r)\|_2^2\,dr}.$$

Applying the definition of $\sigma$, we deduce

$$(2.17)\qquad \mathbb{E}\int_0^\sigma \|\eta_N(t)\|_3^2\,dt \le \|h\|_2^2\exp\left(c(K_0+1)+B_0T\right),$$

which yields Lemma 2.3.

2.3. **Proof of** (2.5)**.**

Let $\psi\in C^\infty(\mathbb{R};[0,1])$ such that

$$\psi = 0 \ \text{ on } \ (K_0+1,\infty) \quad\text{ and }\quad \psi = 1 \ \text{ on } \ (-\infty, K_0).$$

For any process $X$, we set

$$\psi_X = \psi\left(\int_0^T \|X(s)\|_2^2\,ds\right).$$

Remark that

$$(2.18)\qquad \left|\mathbb{E}\left(g(X_N(T,x_0^2)) - g(X_N(T,x_0^1))\right)\right| \le I_0 + |g|_\infty\left(I_1+I_2\right),$$

where

$$\begin{cases} I_0 &=& \left|\mathbb{E}\left(g(X_N(T,x_0^2))\psi_{X_N(\cdot,x_0^2)} - g(X_N(T,x_0^1))\psi_{X_N(\cdot,x_0^1)}\right)\right|,\\ I_i &=& \mathbb{P}\left(\int_0^T\left\|X_N(s,x_0^i)\right\|_2^2 ds > K_0\right).\end{cases}$$

For any $\theta\in[1,2]$, we set

$$\begin{cases} x_0^\theta = (2-\theta)x_0^1 + (\theta-1)x_0^2, & X_\theta = X_N(\cdot,x_0^\theta),\\ \eta_\theta(t) = \eta_N(t,0,x_0^\theta), & \sigma_\theta = \sigma(X_\theta).\end{cases}$$

Recall that $\sigma$ was defined in (2.14). For a better readability, the dependance on $N$ has been omitted. Setting

$$h = x_0^2 - x_0^1,$$

we have

$$I_0 \le \int_1^2 |J_\theta|\, d\theta \quad J_\theta = \left(\nabla \mathbb{E}\left(g(X_\theta(T))\psi_{X_\theta}\right), h\right). \tag{2.19}$$

To bound $J_\theta$, we apply a truncated Bismut-Elworthy-Li formula (See appendix A)

$$J_\theta = \frac{1}{T} J'_{\theta,1} + 2J'_{\theta,2}, \tag{2.20}$$

where

$$\begin{cases} J'_{\theta,1} &= \mathbb{E}\left(g(X_\theta(T))\psi_{X_\theta}\int_0^{\sigma_\theta}(\phi^{-1}(X_\theta(t))\cdot\eta_\theta(t)\cdot h, dW(t))\right), \\ J'_{\theta,2} &= \mathbb{E}\left(g(X_\theta(T))\psi'_{X_\theta}\int_0^{\sigma_\theta}\left(1-\frac{t}{T}\right)(AX_\theta(t), A(\eta_\theta(t)\cdot h))\ dt\right), \\ \psi'_X &= \psi'\left(\int_0^T \|X_\theta(s)\|_2^2\, ds\right). \end{cases}$$

It follows from Hölder inequality that

$$\left|J'_{\theta,2}\right| \le |g|_\infty\, |\psi'|_\infty \sqrt{\mathbb{E}\int_0^{\sigma_\theta}\|X_\theta(t)\|_2^2\ dt}\sqrt{\mathbb{E}\int_0^{\sigma_\theta}\|\eta_\theta(t)\cdot h\|_2^2\ dt}.$$

and from Hypothesis 1.4 that

$$\left|J'_{\theta,1}\right| \le |g|_\infty\, B_0\sqrt{\mathbb{E}\int_0^{\sigma_\theta}\|\eta_\theta(t)\cdot h\|_3^2\ dt}.$$

Hence for any $T \le 1$

$$|J_\theta| \le c(B_0, D)\, |g|_\infty\, \frac{1}{T}\sqrt{\mathbb{E}\int_0^{\sigma_\theta}\|\eta_\theta(t)\cdot h\|_3^2\ dt}. \tag{2.21}$$

Combining (2.21) and Lemma 2.3, we obtain

$$|J_\theta| \le c(B_0, D)\, |g|_\infty\, \frac{\|h\|_2}{T},$$

which yields, by (2.4) and (2.19),

$$I_0 \le c(B_0, D)\, |g|_\infty \sqrt{T}.$$

Since $B_0 T^3 \le B_0 T$, we can apply Lemma 2.2 to control $I_1 + I_2$ in (2.18) if (2.4) holds. Hence (2.5) follows provided (2.4) holds, which yields Proposition 2.1.

## 3. Time of return in a small ball of $\mathbb{H}_2$

Assume that Hypothesis 1.4 holds. Let $N \in \mathbb{N}$ and $T, \delta, Z_1, Z_2$ be as in Proposition 2.1. Let $(W, \widetilde{W})$ be a couple of independant cylindrical Wiener processes on $H$. We denote by $X_N(\cdot, x_0)$ and $\widetilde{X}_N(\cdot, x_0)$ the solutions of (1.3) associated

to $W$ and $\widetilde{W}$, respectively. We build a couple of random variables $(V_1, V_2) = (V_1(x_0^1, x_0^2), V_2(x_0^1, x_0^2))$ on $P_N H$ as follows

$$(3.1)\quad (V_1, V_2) = \begin{cases} (X_N(\cdot, x_0), X_N(\cdot, x_0)) & \text{if } x_0^1 = x_0^2 = x_0, \\ (Z_1(x_0^1, x_0^2), Z_2(x_0^1, x_0^2)) & \text{if } (x_0^1, x_0^2) \in B_{\mathbb{H}_2}(0, \delta) \backslash \{x_0^1 = x_0^2\}, \\ (X_N(\cdot, x_0^1), \widetilde{X}_N(\cdot, x_0^2)) & \text{else,} \end{cases}$$

We then build $(X^1, X^2)$ by induction on $T\mathbb{N}$. Indeed, we first set $X^i(0) = x_0^i$ for $i = 1, 2$. Then, assuming that we have built $(X^1, X^2)$ on $\{0, T, 2T, \ldots, nT\}$, we take $(V_1, V_2)$ as above independent of $(X^1, X^2)$ and we set

$$X^i((n+1)T) = V_i(X^1(nT), X^2(nT)) \quad \text{for } i = 1, 2.$$

It follows that $(X^1, X^2)$ is a discrete strong Markov process and a coupling of $(\mathcal{D}(X_N(\cdot, x_0^1)), \mathcal{D}(X_N(\cdot, x_0^2)))$ on $T\mathbb{N}$. Moreover, if $(X^1, X^2)$ are coupled at time $nT$, then they remain coupled for any time after.

We set

$$(3.2)\qquad \tau = \inf\left\{ t \in T\mathbb{N} \backslash \{0\} \;\middle|\; \left\|X^1(t)\right\|_2^2 \vee \left\|X^2(t)\right\|_2^2 \le \delta \right\}.$$

The aim of this section is to establish the following result.

**Proposition 3.1.** *Assume that Hypothesis 1.4 holds. There exist* $\delta^3 = \delta^3(B_0, D, \varepsilon, \delta)$, $\alpha = \alpha(\phi, D, \varepsilon, \delta) > 0$ *and* $K" = K"(\phi, D, \varepsilon, \delta)$ *such that for any* $(x_0^1, x_0^2) \in H \times H$ *and any* $N \in \mathbb{N}$

$$\mathbb{E}\left(e^{\alpha\tau}\right) \le K" \left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right),$$

*provided* $\|f\|_\varepsilon^2 \le \delta^3$.

The result is based on the fact that, in the absence of noise and forcing term, all solutions go to zero exponentially fast in $H$. A similar idea is used for the two-dimensional Navier-Stokes equations in [23]. The proof is based on the following four Lemmas. The first one allows to control the probability that the contribution of the noise is small. Its proof strongly uses the assumption that the noise is diagonal in the eigenbasis of $A$. As already mentioned, in the additive case, the proof is easy and does not need this assumption.

**Lemma 3.2.** *Assume that Hypothesis 1.4 holds. For any* $t, M > 0$*, there exists* $p_0(t, M) = p_0(t, M, \varepsilon, (|\phi_n|_\infty)_n, D) > 0$ *such that for any adapted process* $X$

$$\mathbb{P}\left(\sup_{(0,t)} \|Z\|_2^2 \le M\right) \ge p_0(t, M),$$

*where*

$$Z(t) = \int_0^t e^{-A(t-s)} \phi(X(s)) dW(s).$$

It is proved in section 3.1.

Then, using this estimate and the smallness assumption on the forcing term, we estimate the moment of the first return time in a small ball in $H$.

Let $\delta_3 > 0$. We set

$$\tau_{L^2} = \tau \wedge \inf\left\{ t \in T\mathbb{N}^* \;\middle|\; \left|X^1(t)\right|^2 \vee \left|X^2(t)\right| \ge \delta_3 \right\}.$$

**Lemma 3.3.** *Assume that Hypothesis 1.4 holds. Then, for any $\delta_3 > 0$, there exist $C_3(\delta_3)$, $C_3'(\delta_3)$ and $\gamma_3(\delta_3)$ such that for any $(x_0^1, x_0^2) \in (\mathbb{H}_2)^2$*

$$\mathbb{E}\left(e^{\gamma_3 \tau_{L^2}}\right) \le C_3 \left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right),$$

*provided*

$$|f| \le C_3'.$$

The proof is postponed to section 3.2.

Then, we need to get a finer estimate in order to control the time necessary to enter a ball in stronger topologies. To prove the two next lemmas, we use an argument similar to one used in the determinist theory (see [35], chapter 7).

**Lemma 3.4.** *Assume that Hypothesis 1.4 holds. Then, for any $\delta_4$, there exist $p_4(\delta_4) > 0$, $C_4'(\delta_4) > 0$ and $R_4(\delta_4) > 0$ such that for any $x_0$ verifying $|x_0|^2 \le R_4$, we have for any $T \le 1$*

$$\mathbb{P}\left(\|X_N(T, x_0)\|^2 \le \delta_4\right) \ge p_4,$$

*provided*

$$|f| \le C_4'.$$

The proof is postponed to section 3.3.

**Lemma 3.5.** *Assume that Hypothesis 1.4 holds. Then, for any $\delta_5$, there exist $p_5(\delta_5) > 0$, $C_5'(\delta_5) > 0$ and $R_5(\delta_5) > 0$ such that for any $x_0$ verifying $\|x_0\|^2 \le R_5$ and for any $T \le 1$*

$$\mathbb{P}\left(\|X_N(T, x_0)\|_2^2 \le \delta_5\right) \ge p_5.$$

*provided*

$$\|f\|_\varepsilon \le C_5'.$$

The proof is postponed to section 3.4.

*Proof of Proposition 3.1*: We set

$$\delta_5 = \delta, \quad \delta_4 = R_5(\delta_5), \quad \delta_3 = R_4(\delta_4), \quad p_4 = p_4(\delta_4), \quad p_5 = p_5(\delta_5), \quad p_1 = (p_4 p_5)^2,$$

and

$$\delta^3 = C_3'(\delta_3) \wedge C_4'(\delta_4) \wedge C_5'(\delta_5).$$

By the definition of $\tau_{L^2}$, we have

$$\left|X^1(\tau_{L^2})\right|^2 \vee \left|X^2(\tau_{L^2})\right|^2 \le R_4(\delta_4).$$

We distinguish three cases.

The first case is $\left\|X^1(\tau_{L^2})\right\|_2^2 \vee \left\|X^2(\tau_{L^2})\right\|_2^2 \le \delta$, which obviously yields

$$\mathbb{P}\left(\min_{k=0,\dots,2} \max_{i=1,2} \left\|X^i(\tau_{L^2} + kT)\right\|_2^2 \le \delta \mid \left(X^2(\tau_{L^2}), X^2(\tau_{L^2})\right)\right) \ge p_1. \tag{3.3}$$

We now treat the case $x_0 = X^1(\tau_{L^2}) = X^2(\tau_{L_2})$ with $\|x_0\|_2^2 > \delta$. Combining Lemma 3.4 and Lemma 3.5, we deduce from the weak Markov property of $X_N$ that

$$\mathbb{P}\left(\|X_N(2T, x_0)\|_2^2 \le \delta\right) \ge p_5 p_4,$$

provided $|x_0|^2 \le R_4$. Recall that, in that case, $X^1(\tau_{L^2}+2T) = X^2(\tau_{L^2}+2T)$. Hence, since the law of $X^1(\tau_{L^2}+2T)$ conditioned by $(X^1(\tau_{L^2}), X^2(\tau_{L_2}))$ is $\mathcal{D}(X_N(2T, x_0))$, it follows

$$\mathbb{P}\left(\max_{i=1,2} \left\|X^i(\tau_{L^2}+2T)\right\|_2^2 \le \delta \mid \left(X^1(\tau_{L^2}), X^2(\tau_{L_2})\right)\right) \ge p_4 p_5 \ge p_1,$$

and then (3.3)

The last case is $X^1(\tau_{L^2}) \ne X^2(\tau_{L_2})$ and $\left\|X^1(\tau_{L^2})\right\|_2^2 \vee \left\|X^2(\tau_{L^2})\right\|_2^2 > \delta$. In that case, $(X^1(\tau_{L^2}+T), X^2(\tau_{L^2}+T))$ conditioned by $(X^1(\tau_{L^2}), X^2(\tau_{L_2}))$ are independent. Hence, since the law of $X^i(\tau_{L^2}+T)$ conditioned by $(X^1(\tau_{L^2}), X^2(\tau_{L_2})) = (x_0^1, x_0^2)$ is $\mathcal{D}(X_N(T, x_0^i))$, it follows from Lemma 3.4 that

$$\mathbb{P}\left(\max_{i=1,2} \left\|X^i(\tau_{L^2}+T)\right\|_1^2 \le \delta_4 \mid \left(X^1(\tau_{L^2}), X^2(\tau_{L_2})\right)\right) \ge p_4^2.$$

Then, we distinguish the three cases $(\left\|X^i(\tau_{L^2}+T)\right\|_1^2)_{i=1,2}$ in the small ball of $\mathbb{H}_2$, equal or different and we deduce from Lemma 3.5 by the same method

$$\mathbb{P}\left(\min_{k=1,2} \max_{i=1,2} \left\|X^i(\tau_{L^2}+kT)\right\|_2^2 \le \delta \mid \left(X^1(\tau_{L^2}+T), X^2(\tau_{L_2}+T)\right)\right) \ge p_5^2,$$

provided

$$\max_{i=1,2} \left\|X^i(\tau_{L^2}+T)\right\|_1^2 \le \delta_4.$$

Combining the two previous inequalities, we deduce (3.3) for the latter case. We have thus proved that (3.3) is true almost surely.

Integrating (3.3), we obtain

$$\mathbb{P}\left(\min_{k=0,\dots,2} \max_{i=1,2} \left\|X^i(\tau_{L^2}+kT)\right\|_2^2 \le \delta\right) \ge p_1. \tag{3.4}$$

Combining Lemma 3.3 and (3.4), we conclude.

### 3.1. **Probability of having a small noise.**

We now establish Lemma 3.2.

We deduce from Hölder inequality and from $\sum_n \mu_n^{-2} < \infty$ that Hypothesis 1.4 implies the following fact: for any $\varepsilon_0 \in (0, \varepsilon)$, there exists $\alpha \in (0,1)$, a family $(\bar\phi_n)_n$ of measurable maps $H \to \mathbb{R}$ and a family $(b_i)_i$ of positive numbers such that

$$\begin{cases} \phi(x)\cdot e_n = b_n \bar\phi_n(x) e_n, \\ \sup_{x\in H} \left|\bar\phi_n(x)\right| \le 1, \quad B^* = \sum_n \mu_n^{1+\varepsilon_0}\ (b_n)^{2(1-\alpha)} < \infty. \end{cases} \tag{3.5}$$

For simplicity we restrict our attention to the case $t = 1$. The generalization is easy.

Remark that

$$Z(t) = \sum_n b_n Z_n(t) e_n,$$

where

$$Z_n(t) = \int_0^t e^{-\mu_n(t-s)} \bar\phi_n(X(s)) dW_n(s), \quad \text{where } W = \sum_n W_n e_n.$$

It follows from $\|Z\|_2^2=\sum_n b_n^2\mu_n\left|\sqrt{\mu_n}Z_n\right|^2$ and from (3.5) that

$$\mathbb{P}\left(\sup_{(0,1)}\|Z\|_2^2\le B^*M\right)\ge\mathbb{P}\left(\sup_{(0,1)}|\sqrt{\mu_n}Z_n|^2\le M\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\,,\ \forall\, n\right). \tag{3.6}$$

Setting

$$W_n'(t)=\sqrt{\mu_n}W_n\left(\frac{t}{\mu_n}\right),$$

we obtain $(W_n')_n$ a family of independent brownian motions. Moreover we have

$$\sqrt{\mu_n}Z_n(t)=Z_n'(\mu_n t),$$

where

$$Z_n'(t)=\int_0^t e^{-(t-s)}\psi_n(s)dW_n'(s),\quad \psi_n(s)=\bar\phi_n\left(X\left(\frac{s}{\mu_n}\right)\right).$$

Hence, it follows from (3.6) that

$$\mathbb{P}\left(\sup_{(0,1)}\|Z\|_2^2\le B^*M\right)\ge\mathbb{P}\left(\sup_{(0,\mu_n)}|Z_n'|^2\le M\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\,,\ \forall\, n\right). \tag{3.7}$$

Let $W_{n,i}'=W_n'(i+\cdot)-W_n'(i)$ on $(0,1)$. We set

$$M_{n,i}(t)=\begin{cases}0 & \text{if}\quad t\le 0,\\ \int_0^{1\wedge t}e^s\,\psi_n(i+s)dW_{n,i}'(s) & \text{if}\quad t\ge 0.\end{cases}$$

Remark that

$$Z_n'(t)=\sum_{i=1}^{\infty}e^{-(t-i)}M_{n,i}(t-i),$$

which yields for any $q\in\mathbb{N}$

$$\sup_{(0,q)}\|Z_n'\|_2\le\left(\frac{e}{e-1}\right)\max_{i=0,\dots,q-1}\sup_{(0,1)}|M_{n,i}|\,. \tag{3.8}$$

Remark that $(W_{n,k}')_{n,k}$ is a family of independant brownian motions on $(0,1)$. It follows that $(M_{n,k})_{n,k}$ are martingales verifying $\langle M_{n,k},M_{n',k'}\rangle=0$ if $(n,k)\ne(n',k')$.

Hence, combining a Theorem by Dambis, Dubins and Schwartz (Theorem 4.6 page 174 of [20]) and a Theorem by Knight (Theorem 4.13 page 179 of [20]), we obtain a family $(B_{n,k})_{n,k}$ of independent brownian motions verifying

$$M_{n,k}(t)=B_{n,k}(\langle M_{n,k}\rangle(t)). \tag{3.9}$$

**Remark 3.6.** *In the two previous Theorem (Theorem 4.6 page 174 and Theorem 4.13 page 179 of* [20]*), it is assumed that $\mathbb{P}(<M>(\infty)<\infty)=0$. However, as explained in Problem 4.7 of* [20]*, the proof is easily adapted to the case $\mathbb{P}(<M>(\infty)<\infty)>0$.*

Remarking that for any $t\in(0,1)$

$$\langle M_{n,k}\rangle(t)=\int_0^t|\psi_n(k+s)|^2\,ds\le 1,$$

we deduce from (3.8) and (3.9) that for any $q \in \mathbb{N}^*$

$$\sup_{(0,q)} \|Z_n'\|_2 \leq \left(\frac{e}{e-1}\right) \max_{i=0,\dots,q-1} \sup_{(0,1)} |B_{n,i}|\,.$$

Hence it follows from (3.7) that

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq \mathbb{P}\left(\sup_{(0,1)} |B_{n,i}|^2 \leq cM\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\,,\ \forall\, n,\ \forall\, i \leq \mu_n + 1\right),$$

where $c = \left(\frac{e-1}{e}\right)^2 \frac{1}{B^*}$.
We deduce from the independence of $(B_{n,k})_{n,k}$ that

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq \prod_{n\in\mathbb{N}^*} \left(P\left(cM\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\right)^{\mu_n+1}\right), \tag{3.10}$$

where

$$P(d_0) = \mathbb{P}\left(\sup_{(0,1)} |B_{1,1}|^2 \leq d_0\right).$$

Recall there exists a family $(c_p)_p$ such that

$$\mathbb{E}\left(\sup_{(0,1)} |B_{1,1}|^{2p}\right) \leq c_p.$$

It follows from Chebyshev inequality and from $1 - x \geq e^{-ex}$ for any $x \leq e^{-1}$ that for any $d_0 \leq d_p = \left(e^{-1}c_p\right)^{\frac{1}{p}}$

$$P(d_0) \geq 1 - c_p d_0^{-p} \geq e^{-ec_p d_0^{-p}}.$$

Applying (3.10), we obtain for any $p > 0$

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq C_p(M) \exp\left(-\frac{c_p'}{M^p} \sum_{n>\mathbb{N}(p,M)} \left(\frac{\mu_n+1}{\mu_n^{\varepsilon_0 p}}\right) b_n^{2\alpha p}\right), \tag{3.11}$$

where

$$\begin{cases} N(p,M) &=& \sup\left\{n \in \mathbb{N}\backslash\{0\}\ \middle|\ M\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha} \leq d_p\right\}, \\ C_p(M) &=& \Pi_{n\leq\mathbb{N}(p,M)}\left(P\left(cM\,\mu_n^{\varepsilon_0}\,(b_n)^{-2\alpha}\right)^{\mu_n+1}\right). \end{cases}$$

Choosing $p$ sufficiently high, we deduce from **H0** that

$$\sum_n \left(\frac{\mu_n+1}{\mu_n^{\varepsilon_0 p}}\right) b_n^{2\alpha p} \leq C_p' < \infty,$$

which yields, by (3.11), that for any $M > 0$ and for $p$ sufficiently high

$$\mathbb{P}\left(\sup_{(0,1)} \|Z\|_2^2 \leq M\right) \geq C_p(M) \exp\left(-\frac{c_p{}^{"}}{M^p}\right), \tag{3.12}$$

Remark that for any $p$, $\varepsilon_0$ we have $N(p,M) < \infty$. Moreover, it is well-known that for any $d_0 > 0$, $P(d_0) > 0$, which yields $C_p(M) > 0$ and then Lemma 3.2.

3.2. **Proof of Lemma 3.3.**
For simplicity in the redaction, we restrict our attention to the case $f = 0$. The generalisation is easy.

Recall (1.5)

$$\mathbb{E}\,|X_N(t)|^2 \leq e^{-\mu_1 t}\,|x_0|^2 + \frac{c}{\mu_1}B_0.$$

Since $(X^1, X^2)$ is a coupling of $(\mathcal{D}(X_N(\cdot, x_0^1)), \mathcal{D}(X_N(\cdot, x_0^1)))$ on $T\mathbb{N}$, we obtain

$$\mathbb{E}\left(\left|X^1(nT)\right|^2 + \left|X^2(nT)\right|^2\right) \leq e^{-\mu_1 nT}\left(\left|x_0^1\right|^2 + \left|x_0^2\right|^2\right) + 2\frac{c}{\mu_1}B_0. \tag{3.13}$$

Since $(X^1, X^2)$ is a strong Markov process, it can be deduced that there exist $C_6$ and $\gamma_6$ such that for any $x_0 \in H$

$$\mathbb{E}\left(e^{\gamma_6 \tau'_{L^2}}\right) \leq C_6\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right), \tag{3.14}$$

where

$$\tau'_{L^2} = \inf\left\{t \in T\mathbb{N}\backslash\{0\} \;\middle|\; \left|X^1(t)\right|^2 + \left|X^2(t)\right|^2 \geq 4cB_0\right\}.$$

Taking into account (3.14), a standard argument gives that, in order to establish Lemma 3.3, it is sufficient to prove that there exist $(p_7, T_7)$ such that

$$\mathbb{P}\left(|X_N(t, x_0)|^2 \leq \delta_3\right) \geq p_7(\delta_3, t) > 0, \tag{3.15}$$

provided $N \in \mathbb{N}$, $t \geq T_7(\delta_3)$ and $|x_0|^2 \leq 4cB_0$.
We set

$$Z(t) = \int_0^t e^{-A(t-s)}\phi(X_N(s))dW(s), \quad Y_N = X_N - P_N Z, \quad M = \sup_{(0,t)} \|Z\|_2^2.$$

Assume that there exist $M_7(\delta_3) > 0$ and $T_7(\delta_3)$ such that

$$M \leq M_7(\delta_3) \quad \text{implies} \quad |Y_N(t)|^2 \leq \frac{\delta_3}{4}, \tag{3.16}$$

provided $t \geq T_7(\delta_3)$ and $|x_0|^2 \leq 4cB_0$. Then (3.15) results from Lemma 3.2 with

$$M = \min\left\{M_7(\delta_3), \frac{\delta_3}{4}\right\}.$$

We now prove (3.16). Remark that

$$\frac{d}{dt}Y_N + AY_N + P_N B(Y_N + P_N Z) = 0. \tag{3.17}$$

Taking the scalar product of (3.17) with $Y_N$, it follows that

$$\frac{d}{dt}|Y_N|^2 + 2\|Y_N\|^2 = -2(Y_N, B(Y_N + P_N Z)). \tag{3.18}$$

Recalling that $(B(y,x),x) = 0$, we obtain

$$-2(Y_N, B(Y_N + P_N Z)) = -2(Y_N, (Y_N, \nabla)P_N Z) - 2(Y_N, B(P_N Z)).$$

We deduce from Hölder inequalities and Sobolev embedding that

$$-(z, (x, \nabla)y) \leq c\,\|z\|\,\|x\|\,\|y\|\,.$$

Hence it follows from (3.18) that

$$\frac{d}{dt}|Y_N|^2 + 2\|Y_N\|^2 \leq c\,\|Z\|^2\,\|Y_N\| + c\,\|Z\|\,\|Y_N\|^2\,,$$

which yields, by an arithmetico-geometric inequality,

$$\frac{d}{dt}|Y_N|^2 + 2\|Y_N\|^2 \le cM^{\frac{1}{2}}\|Y_N\|^2 + cM^{\frac{3}{2}}.$$

It follows that $M \le \frac{1}{c^2}$ implies

$$\frac{d}{dt}|Y_N|^2 + \|Y_N\|^2 \le cM^{\frac{3}{2}} \quad \text{on } (0,t). \tag{3.19}$$

Integrating, we deduce from $|x_0|^2 \le 4cB_0$ that

$$|Y_N(t)|^2 \le 4ce^{-\mu_1 t}B_0 + c\left(\frac{M^{\frac{3}{2}}}{\mu_1}\right).$$

Choosing $t$ sufficiently large and $M$ sufficiently small we obtain (3.16) which yields (3.15) and then Lemma 3.3.

**Remark 3.7.** *In order to avoid a lengthy proof, we have not splitted the arguments in several cases as in the proof of Proposition 3.1. The reader can complete the details.*

3.3. **Proof of Lemma 3.4.**

We use the decomposition $X_N = Y_N + P_N Z$ defined in section 3.2 and set

$$M = \sup_{(0,T)} \|Z\|_2^2.$$

Integrating (3.19), we obtain for $M$ satisfying the same assumption $M \le \frac{1}{c^2}$

$$\frac{1}{T}\int_0^T \|Y_N(t)\|^2 dt \le \frac{1}{T}|x_0|^2 + cM^{\frac{3}{2}},$$

which yields, by a Chebyshev inequality,

$$\lambda\left(t \in (0,T) \;\middle|\; \|Y_N(t)\|^2 \le \frac{2}{T}|x_0|^2 + 2cM^{\frac{3}{2}}\right) \ge \frac{T}{2}, \tag{3.20}$$

where $\lambda$ denotes the Lebesgue measure on $(0,T)$.

Setting

$$\tau_{\mathbb{H}_1} = \inf\left\{t \in (0,T) \;\middle|\; \|Y_N(t)\|^2 \le \frac{2}{T}|x_0|^2 + 2cM^{\frac{3}{2}}\right\},$$

we deduce from (3.20) and the continuity of $Y_N$ that

$$\|Y_N(\tau_{\mathbb{H}_1})\|^2 \le \frac{2}{T}|x_0|^2 + 2cM^{\frac{3}{2}}. \tag{3.21}$$

Taking the scalar product of $2AY$ and (3.17), we obtain

$$\frac{d}{dt}\|Y_N\|^2 + 2\|Y_N\|_2^2 = -2(AY_N, B(Y_N + P_N Z)). \tag{3.22}$$

It follows from Hölder inequalities, Sobolev Embeddings and Agmon inequality that

$$-2(Ay, \widetilde{B}(x,z)) \le c\|y\|_2 \|z\|^{\frac{1}{2}} \|z\|_2^{\frac{1}{2}} \|x\|,$$

where $\widetilde{B}(x,y) = (x,\nabla)y + (y,\nabla)x$. Hence, we obtain by applying arithmetico-geometric inequalities

$$\begin{cases} -2(AY_N, B(Y_N)) & \leq & c\,\|Y_N\|_2^{\frac32}\,\|Y_N\|^{\frac32} & \leq & \frac14\,\|Y_N\|_2^2 + c\,\|Y_N\|^6, \\ -2(AY_N, B(P_N Z)) & \leq & c\,\|Y_N\|_2\,\|Z\|^{\frac32}\,\|Z\|_2^{\frac12} & \leq & \frac14\,\|Y_N\|_2^2 + c\,\|Z\|_2^4, \\ -2(AY_N, \widetilde{B}(Y_N, P_N Z)) & \leq & c\,\|Y_N\|_2^{\frac32}\,\|Y_N\|^{\frac12}\,\|Z\| & \leq & c\,\|Z\|\,\|Y_N\|_2^2 \end{cases}$$

Remarking that $B(Y_N + P_N Z) = B(Y_N) + \widetilde{B}(Y_N, P_N Z) + B(P_N Z)$, it follows from (3.22) that $M \leq \frac{1}{4c}$ implies

$$\frac{d}{dt}\,\|Y_N\|^2 + \|Y_N\|_2^2 \leq c\,\|Y_N\|^2\left(\|Y_N\|^4 - 4K_0^2\right) + cM^2, \tag{3.23}$$

where $K_0$ is defined in (2.10). Let us set

$$\sigma_{\mathbb{H}_1} = \inf\left\{t \in (\tau_{\mathbb{H}_1}, T)\ \middle|\ \|Y_N(t)\|^2 > 2K_0\right\},$$

and remark that on $(\tau_{\mathbb{H}_1}, \sigma_{\mathbb{H}_1})$, we have

$$\frac{d}{dt}\,\|Y_N\|^2 + \|Y_N\|_2^2 \leq cM^2. \tag{3.24}$$

Integrating, we obtain that

$$\|Y_N(\sigma_{\mathbb{H}_1})\|^2 + \int_{\tau_{\mathbb{H}_1}}^{\sigma_{\mathbb{H}_1}} \|Y_N(t)\|_2^2\,dt \leq \|Y_N(\tau_{\mathbb{H}_1})\|^2 + cM^2. \tag{3.25}$$

Combining (3.21) and (3.25), we obtain that, for $M$ and $|x_0|^2$ sufficiently small,

$$\|Y_N(\sigma_{\mathbb{H}_1})\|^2 \leq \frac{\delta_4}{4} \wedge K_0,$$

which yields $\sigma_{\mathbb{H}_1} = T$. It follows that

$$\|X_N(T)\|^2 \leq \delta_4, \tag{3.26}$$

provided $M$ and $|x_0|^2$ sufficiently small. It remains to use Lemma 3.2 to get Lemma 3.4.

### 3.4. **Proof of Lemma 3.5.**

It follows from (3.24) that

$$\int_0^T \|Y_N(t)\|_2^2\,dt \leq \|x_0\|^2 + cM^2,$$

provided $M \leq \frac{1}{4c}$ and $\|x_0\|^2 + cM^2 \leq K_0$.
Applying the same argument as in the previous subsection, it is easy to deduce that there exists a stopping times $\tau_{\mathbb{H}_2} \in (0,T)$ such that

$$\|Y_N(\tau_{\mathbb{H}_2})\|_2^2 \leq \frac{2}{T}\left(\|x_0\|^2 + cM^2\right), \tag{3.27}$$

provided $M$ and $\|x_0\|$ are sufficiently small.

Taking the scalar product of (3.17) and $2A^2 Y_N$, we obtain

$$\frac{d}{dt}\,\|Y_N\|_2^2 + 2\,\|Y_N\|_3^2 = -2\left(A^{\frac32}Y_N, A^{\frac12}B(Y_N + P_N Z)\right). \tag{3.28}$$

Applying Hölder inequality, Sobolev Embeddings $\mathbb{H}_2 \subset L^\infty$ and $\mathbb{H}_1 \subset L^4$ and arithmetico-geometric inequality, we obtain

$$-2\left(A^{\frac{3}{2}}y, A^{\frac{1}{2}}B(x,y)\right) \le c\,\|y\|_3\,\|x\|_2\,\|y\|_2 \le \frac{1}{4}\,\|y\|_3^2 + c\left(\|x\|_2^4 + \|y\|_2^4\right).$$

Hence we deduce from (3.28) and from $B(Y_N + P_N Z) = B(Y_N) + B(Y_N, P_N Z) + B(P_N Z)$

$$\frac{d}{dt}\,\|Y_N\|_2^2 + \|Y_N\|_3^2 \le c\,\|Y_N\|_2^2\,(\|Y_N\|_2^2 - 2K_1) + c\,\|Z\|_2^4, \tag{3.29}$$

where $K_1$ is defined as $K_0$ in (2.10) but with a different $c$. We set

$$\sigma_{\mathbb{H}_2} = \inf\left\{t \in (\tau_{\mathbb{H}_2}, T)\ \middle|\ \|Y_N(t)\|_2^2 > 2K_1\right\},$$

Integrating (3.29), we obtain

$$\|Y_N(\sigma_{\mathbb{H}_2})\|_2^2 + \int_{\tau_{\mathbb{H}_2}}^{\sigma_{\mathbb{H}_2}} \|Y_N(t)\|_3^2\,dt \le \|Y_N(\tau_{\mathbb{H}_2})\|_2^2 + cM^2.$$

Taking into account (3.27) and choosing $\|x_0\|^2$ and $M^2$ sufficiently small, we obtain

$$\|Y_N(\sigma_{\mathbb{H}_2})\|_2^2 \le \frac{\delta}{4} \wedge K_1.$$

It follows that $\sigma_{\mathbb{H}_2} = T$ and that

$$\|X_N(T)\|^2 \le \delta, \tag{3.30}$$

provided $M$ and $\|x_0\|$ sufficiently small, which yields (2.2).

## 4. Proof of Theorem 1.8

As already explained, Theorem 1.8 follows from Proposition 1.15. We now prove Proposition 1.15. Let $(x_0^1, x_0^2) \in (\mathbb{H}_2)^2$. Let us recall that the process $(X^1, X^2)$ is defined at the beginning of section 3.

Let $\delta > 0$, $T \in (0,1)$ be as in Proposition 2.1 and $\tau$ defined in (3.2), setting

$$\tau_1 = \tau, \quad \tau_{k+1} = \inf\left\{t > \tau_k\ \middle|\ \left\|X^1(t)\right\|_2^2 \vee \left\|X^2(t)\right\|_2^2 \le \delta\right\}.$$

it can be deduced from the strong Markov property of $(X^1, X^2)$ and from Proposition 3.1 that

$$\mathbb{E}\left(e^{\alpha\tau_{k+1}}\right) \le K"\mathbb{E}\left(e^{\alpha\tau_k}\left(1 + \left|X^1(\tau_k)\right|^2 + \left|X^2(\tau_k)\right|^2\right)\right),$$

which yields, by the Poincaré inequality,

$$\begin{cases} \mathbb{E}\left(e^{\alpha\tau_{k+1}}\right) & \le & cK"(1+2\delta)\mathbb{E}\left(e^{\alpha\tau_k}\right), \\ \mathbb{E}\left(e^{\alpha\tau_1}\right) & \le & K"\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right). \end{cases}$$

It follows that there exists $K > 0$ such that

$$\mathbb{E}\left(e^{\alpha\tau_k}\right) \le K^k\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right).$$

Hence, applying Jensen inequality, we obtain that, for any $\theta \in (0,1)$

$$\mathbb{E}\left(e^{\theta\alpha\tau_k}\right) \le K^{\theta k}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right). \tag{4.1}$$

We deduce from Proposition 2.1 and from (3.1) that

$$\mathbb{P}\left(X^1(T) \neq X^2(T)\right) \leq \frac{1}{4},$$

provided $(x_0^1, x_0^2)$ are in the ball of $(\mathbb{H}^2)^2$ with radius $\delta$.

Setting

$$k_0 = \inf\left\{k \in \mathbb{N} \;\middle|\; X^1(\tau_k + T) = X^2(\tau_k + T)\right\},$$

it follows that $k_0 < \infty$ almost surely and that

$$\mathbb{P}\left(k_0 > n\right) \leq \left(\frac{1}{4}\right)^n. \tag{4.2}$$

Let $\theta \in (0,1)$. We deduce from Schwartz inequality that

$$\mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_{k_0}}\right) = \sum_{n=1}^{\infty} \mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_n} 1_{k_0=n}\right) \leq \sum_{n=1}^{\infty} \sqrt{\mathbb{P}\left(k_0 \geq n\right) \mathbb{E}\left(e^{\theta\alpha\tau_n}\right)}.$$

Combining (4.1) and (4.2), we deduce

$$\mathbb{E}\left(e^{\frac{\theta}{2}\alpha\tau_{k_0}}\right) \leq \left(\sum_{n=0}^{\infty}\left(\frac{K^\theta}{2}\right)^n\right)\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right).$$

Hence, choosing $\theta \in (0,1)$ sufficiently small, we obtain that there exists $\gamma > 0$ non depending on $N \in \mathbb{N}$ such that

$$\mathbb{E}\left(e^{\gamma\tau_{k_0}}\right) \leq 4\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right). \tag{4.3}$$

Recall that if $(X^1, X^2)$ are coupled at time $t \in T\mathbb{N}$, then they remain coupled for any time after. Hence $X^1(t) = X^2(t)$ for $t > \tau_{k_0}$. It follows

$$\mathbb{P}\left(X^1(nT) \neq X^2(nT)\right) \leq 4e^{-\gamma nT}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right).$$

Since $(X^1(nT), X^2(nT))$ is a coupling of $((\mathcal{P}_{nT}^N)^*\delta_{x_0^1}, (\mathcal{P}_{nT}^N)^*\delta_{x_0^2})$, we deduce from Lemma 1.14

$$\left\|\left(\mathcal{P}_{nT}^N\right)^*\delta_{x_0^1} - \left(\mathcal{P}_{nT}^N\right)^*\delta_{x_0^2}\right\|_{var} \leq 4e^{-\gamma nT}\left(1 + \left|x_0^1\right|^2 + \left|x_0^2\right|^2\right), \tag{4.4}$$

for any $n \in \mathbb{N}$ and any $(x_0^1, x_0^2) \in (\mathbb{H}_2)^2$.

Recall that the existence of an invariant measure $\mu_N \in P(P_N H)$ is justified in section 1.3. Let $\lambda \in P(H)$ and $t \in \mathbb{R}^+$. We set $n = \lfloor \frac{t}{T} \rfloor$ and $C = 4e^{\gamma T}$. Integrating $(x_0^1, x_0^2)$ over $((\mathcal{P}_{t-nT}^N)^*\lambda) \otimes \mu_N$ in (4.4), we obtain

$$\left\|\left(\mathcal{P}_t^N\right)^*\lambda - \mu_N\right\|_{var} \leq Ce^{-\gamma t}\left(1 + \int_H \left|x_0\right|^2 \lambda(dx)\right),$$

which establishes (1.19).

## Appendix A. Proof of (2.20)

For simplicity in the redaction, we omit $\theta$ and $N$ in our notations.

Remark that

$$J = \left(\nabla\mathbb{E}\left(g(X(T))\psi_X\right), h\right) = J_1 + 2J_2, \tag{A.1}$$

where

$$\begin{cases} J_1 &= \mathbb{E}\left( (\nabla g(X(T)), \eta(T,0)\cdot h)\, \psi_X \right), \\ J_2 &= \mathbb{E}\left( g(X(T))\psi_X' \int_0^T (AX(t), A(\eta(t,0)\cdot h))\, ds \right). \end{cases}$$

According to [29], let us denote by $D_s F$ the Malliavin derivative of $F$ at time $s$. We have the following formula of the Malliavin derivative of the solution of a stochastic differential equation

$$D_s X(t) = 1_{t\geq s}\eta(t,s)\cdot\phi(X(s)),$$

which yields

$$\int_0^t D_s X(t)\cdot m(s)\, ds = G(t)\cdot m, \tag{A.2}$$

where

$$G(t)\cdot m = \int_0^t \eta(t,s)\cdot\phi(X(s))\cdot m(s)\, ds.$$

The uniqueness of the solutions gives

$$\eta(t,0)\cdot h = \eta(t,s)\cdot(\eta(s,0)\cdot h) \text{ for any } 0\leq s\leq t,$$

which yields

$$\eta(T,0)\cdot h = \frac{1}{T}\int_0^T \eta(t,s)\cdot(\eta(s,0)\cdot h)\, ds.$$

Setting

$$w(s) = \phi^{-1}(X(s))\cdot\eta(s,0)\cdot h,$$

we infer from (A.2)

$$\eta(T,0)\cdot h = \frac{1}{T}\, G(T)\cdot w = \frac{1}{T}\int_0^T D_s X(T)\cdot w\, ds, \tag{A.3}$$

which yields

$$(\nabla g(X(T)), \eta(T,0)\cdot h) = \frac{1}{T}\int_0^T (\nabla g(X(T)), D_s X(T)\cdot w)\, ds.$$

Remark that

$$(D_s g(X(T)), w) = (\nabla g(X(T)), D_s X(T)\cdot w).$$

It follows

$$(\nabla g(X(T)), \eta(T,0)\cdot h) = \frac{1}{T}\int_0^T (D_s g(X(T)), w)\, ds,$$

which yields

$$J_1 = \frac{1}{T}\mathbb{E}\int_0^T \psi_X\, (D_s g(X(T)), w)\, ds. \tag{A.4}$$

Recall that the Skohorod integral is the dual operator of the Malliavin derivative (See [29]). It follows

$$J_1 = \frac{1}{T}\mathbb{E}\left( g(X(T))\int_0^T \psi_X(w(t), dW(t)) \right). \tag{A.5}$$

Recall the formula of integration of a product

$$\int_0^T \psi_X(w(t), dW(t)) = \psi_X\int_0^T (w(t), dW(t)) - \int_0^T (D_s\psi_X, w(s))\, ds. \tag{A.6}$$

Remark that

$$D_s\psi_X = 2\psi_X' \int_0^T AD_sX(t)\cdot(AX(t))\,dt,$$

which yields, by $(AX(t), AD_sX(t)\cdot w(s)) = (w(s), AD_sX(t)\cdot(AX(t)))$,

$$\int_0^T (D_s\psi_X, w(s))\,ds = 2\psi_X' \int_0^T\int_0^T (AX(t), AD_sX(t)\cdot w(s))\ dtds.$$

We deduce from (A.3) that

$$\int_0^T (D_s\psi_X, w(s))\,ds = 2\psi_X' \int_0^T t\,(AX(t), A\eta(t,0)\cdot h)\ dt. \tag{A.7}$$

Remark that

$$\psi_X = \psi_X' = 0 \quad \text{if} \quad \sigma < T.$$

Hence combining (A.6) and (A.7), we obtain

$$\int_0^T \psi_X(w(t), dW(t)) = \psi_X \int_0^\sigma (w(t), dW(t)) - 2\psi_X' \int_0^\sigma t\,(AX(t), A\eta(t,0)\cdot h)\ dt.$$

Thus, (2.20) follows from (A.1) and (A.5).

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